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Complete Monotonicity Properties of Certain Approximation Operators*

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INTRODUCTION

The analysis of the properties of approximants induced by properties of the functions they approximate has aroused much interest in the literature. In particular, convexity preserving operators and variation diminishing transformations (see e.g. [3, 10]) fit into this category.

In the same vein, it was observed by Averbach (see [10]) that $B_n(f; x) \geqslant B_{n+1}(f; x)$ for all f convex on [0, 1], where $B_n(f; x)$ is the nth Bernstein polynomial. This observation was extensively generalized by Marshall and Proschan [9] who used majorization techniques to obtain similar results for a wide class of positive linear approximation methods generated through probabilistic considerations. A converse to this type of result was later obtained in [11], where it was proved that the relations

$$L_n(f; x) \geqslant L_{n+1}(f; x) \qquad \text{for all } n, x$$

where $\{L_n\}$ belong to a wide class of positive linear approximation operators, characterize convex functions.

In a colloquium talk where I presented these results, Professor R. Askey raised the following question:

Let Δ_n^r be the rth forward difference operator with respect to n, and let D^r denote the rth derivative with respect to x (with $r = 0$ given the usual interpretation). Then the previous results can be stated as

$$D^0 f(x) \geqslant 0 \quad \text{for all } x \Leftrightarrow \Delta_n{}^0 L_n(f; x) \geqslant 0 \quad \text{for all } n, x, \tag{1}$$

$$D^2 f(x) \geqslant 0 \quad \text{for all } x \Leftrightarrow \Delta_n{}^1 L_n(f; x) \leqslant 0 \quad \text{for all } n, x, \tag{2}$$

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for a class of positive linear operators including the Bernstein polynomial operators. Is it true, then, that

$$D^4 f(x) \geqslant 0 \qquad \text{for all } x \Leftrightarrow \Delta_n{}^2 L_n(f; x) \geqslant 0 \qquad \text{for all } n, x \tag{3}$$

or, in general, that

$$D^{2k} f(x) \geqslant 0 \qquad \text{for all } x \Leftrightarrow (-1)^k\, \Delta_n{}^k L_n(f; x) \geqslant 0 \qquad \text{for all } n, x. \tag{4}$$

The answer to this question is negative even for $k = 2$, as will be established in Section 1. The natural question that arises next is whether an intersection of cones of functions satisfying differential inequalities will suffice.

Arama and Ripianu [1] proved that if $f(x) - f(0)$ is absolutely monotone then $\Delta_n{}^2 B_n(f; x) \geqslant 0$, and Horová [2] derived the same conclusion for f whose first four derivatives are nonnegative.

We will discuss in this note a more symmetric implication, namely, the implication from an infinite collection of conditions on $f(x)$ to an infinite collection of conditions on the sequence of approximants. Specifically, we prove that for a large class of approximants (not including, strangely enough, the Bernstein polynomial operators), absolute monotonicity of the function f implies complete monotonicity of the sequence $\{L_n(f; x)\}_{n=1}^{\infty}$ for each fixed x.

1. A Negative Answer

We prove in this section that the answer to Askey's original question is negative, and in fact relations of type (1), (2), (3) cannot be valid together for nondegenerate approximation operators.

THEOREM 1. *Let $\{L_n\}_{n=1}^{\infty}$ be a sequence of positive linear operators from $C[0, 1]$ to $C[0, 1]$, such that*

$$\lim_{n\to\infty} L_n(f; x) = f(x), \qquad \textit{uniformly} \tag{5}$$

for all $f \in C[0, 1]$. When

$$\Delta_n{}^1 L_n(f; x) \leqslant 0 \qquad \textit{for all } n \textit{ and all } f \textit{ convex} \tag{6}$$

and

$$\Delta_n{}^2 L_n(f; x) \geqslant 0 \qquad \textit{for all } n \textit{ and all } f \textit{ such that } f'' \textit{ is convex,} \tag{7}$$

then

$$L_n(f; x) \equiv f(x) \qquad \textit{for all } n \textit{ and all } f \in C[0, 1].$$

Proof. Set, for all n,

$$U_n = L_n - L_{n+1}.$$

Since $\pm t$ and ± 1 are convex functions, (6) implies that

$$U_n(t; x) \equiv 0 \qquad \text{for all } n, \tag{8}$$

$$U_n(1; x) \equiv 0 \qquad \text{for all } n. \tag{9}$$

Relation (9) implies that $L_n(1; x)$ is independent of n. Making use of (5) we conclude that

$$L_n(1; x) \equiv 1 \qquad \text{for all } n. \tag{10}$$

Returning to (8) and using a similar argument, we deduce that

$$L_n(t; x) \equiv x \qquad \text{for all } n. \tag{11}$$

Let now $V_n = U_n - U_{n+1} = \Delta_n{}^2 L_n$. Since $\pm t^2$ satisfy the requirement that the second derivative be convex, relation (7) implies that

$$V_n(t^2; x) \equiv 0, \tag{12}$$

i.e., that $U_n(t^2; x)$ is independent of n. Thus, $L_n(t^2; x)$ is a solution of a difference equation of the form

$$L_{n+1}(t^2; x) - L_n(t^2; x) \equiv g(x), \tag{13}$$

where $g(x)$ is some continuous function.

Appealing now to (5), we conclude that $g(x) \equiv 0$ and that

$$L_n(t^2; x) \equiv L_1(t^2; x) \equiv x^2. \tag{14}$$

Relations (10), (11), and (14) imply by Korovkin's classical theorem (see [8] p. 14) that $L_n(f; x) \equiv f(x)$, for all n and all $f \in C[0, 1]$.

2. The Absolutely Monotone Case

We start this section by recalling some information about absolutely monotone functions (see, e.g., [7]).

An inequality of the type

$$(-1)^k \Delta_n{}^k B_n(f; x) \geqslant 0 \tag{15}$$

is linear. By making use of the extreme ray structure of the cone $\mathscr{C}_A[0, 1]$ of absolutely monotone functions on [0, 1] we conclude that (15) holds for each $f \in \mathscr{C}_A$ if and only if it holds for $f(t) \equiv t^s$, $s = 0, 1, \ldots$, i.e., if and only if

$$(-1)^k \Delta_n{}^k B_n(t^s; x) \geqslant 0, \qquad s = 0, 1, \ldots \tag{16}$$

A similar result is valid for a subclass of $\mathscr{C}_A[0, \infty)$, where the growth of f is suitably restricted. These observations are simple consequences of the standard limit theorems.

We discuss first the Bernstein polynomial operators. Surprisingly, these operators which serve as standard examples satisfying most of the elegant properties of approximants, fail to do so in this case. In fact, we easily obtain

THEOREM 2. *For $f \in C[0, 1]$ let $B_n(f; x) = \sum_{i=0}^n f(i/n)\binom{n}{i} x^i(1 - x)^{n-i}$ be the* n*th Bernstein polynomial operator. There exists an absolutely monotone function f and a point $x_0 \in [0, 1]$ such that $\{B_n(f, x_0)\}_{n=1}^\infty$ is not completely monotone.*

Proof. Consider the polynomial

$$B_n(t^3; x) = \sum_{i=0}^n \left(\frac{i}{n}\right)^3 \binom{n}{i} x^i(1 - x)^{n-i}.$$

Simple computations transform it into

$$\begin{aligned} B_n(t^3; x) &= x^3 + \frac{x(2x^2 - 3x + 1)}{n^2} + \frac{3x(x - x^2)}{n} \\ &= x^3 + x(1 - x)\left(\frac{3x}{n} + \frac{1 - 2x}{n^2}\right). \end{aligned}$$

Consider now the function

$$f(\alpha) = \frac{3x}{\alpha} + \frac{1 - 2x}{\alpha^2}$$

A direct differentiation yields

$$\begin{aligned} f^{(k)}(\alpha) &= \frac{3x(-1)^k\, k!}{\alpha^{k+1}} + \frac{(1 - 2x)(-1)^k\,(k + 1)!}{\alpha^{k+2}} \\ &= \frac{(-1)^k\, k!}{\alpha^{k+2}} \{(k + 1) + x[3\alpha - 2(k + 1)]\}. \end{aligned}$$

It is easily seen that for k such that $3\alpha < 2(k + 1)$, the expression in the brackets attains its minimum at $x = 1$. This minimum is $3\alpha - (k + 1)$ and is therefore negative if $3\alpha < (k + 1)$. Hence $f(\alpha)$ is not completely monotone.

Taking our lead from this fact, we verify by direct computation that $(1 - x)^{-1} \Delta_n^{10} B_n t^3; x)|_{x=1} < 0$. By continuity, for x_0 near 1, we have $\Delta_n^{10} B(t^3; x_0) < 0$, so that $B_n(t^3; x_0)$ is not completely monotone. Q.E.D.

We turn next to the positive answers. We start with the Szasz–Mirakyan operators, $M_n(f; x)$, constructed through the Poisson distribution. These are defined by

$$M_n(f; x) = e^{-nx} \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) \frac{(nx)^k}{k!}. \tag{17}$$

THEOREM 3. *Let f be absolutely monotone on $[0, \infty)$ and assume there exist constants A, B such that*

$$| f(t)| < Ae^{Bt}, \qquad t \in [0, \infty). \tag{18}$$

Then $\{M_n(f; x)\}_1^\infty$ is a completely monotone sequence for all x.

Proof. Condition (18) assures $M_n(f; x) \to f(x)$, where the convergence is uniform on compact subsets (see [6, p. 333]), and the sufficiency of checking for the extreme rays t^s, $s = 0, 1,\ldots$. The case $s = 0$ is trivial. We now proceed to prove that $\{M_n(t^s; x)\}_{n=1}^\infty$ is completely monotone for $s \geqslant 1$.

We need the following simple result:

Let $s \geqslant 1$, and let a_{js}, $j = 1,\ldots, s$ be defined by

$$t^s = \sum_{j=1}^{s} a_{js} t(t - 1) \cdots (t - j + 1) \tag{19}$$

Then $a_{js} > 0$ for $j = 1,\ldots, s$. (20)

Indeed, if we define $\binom{t}{j} = t(t - 1) \cdots (t - j + 1)/j!$ (where $\binom{t}{0} = 1$) we have

$$\Delta_t \binom{t}{j} = \binom{t+1}{j} - \binom{t}{j} = \binom{t}{j-1}.$$

Hence, for all $1 \leqslant i \leqslant s$

$$\Delta_t^i[t^s] = \sum_{j=1}^{s} j!\, a_{js} \binom{t}{j-i}$$

so that

$$\Delta_t^i[t^s]|_{t=0} = i!\, a_{is}, \qquad 1 \leqslant i \leqslant s.$$

Since $\Delta_t^i[t^s]|_{t=0}$ has the sign of the ith derivative of t^s at some point between 0 and i, (20) follows.

Straightforward computation yields now

$$M_n(t^s; x) = e^{-nx} \sum_{k=0}^{\infty} \left(\frac{k}{n}\right)^s \frac{(nx)^k}{k!} = e^{-nx} \sum_{k=1}^{\infty} \left(\frac{k}{n}\right)^s \frac{(nx)^k}{k!}$$

$$= \frac{e^{-nx}}{n^s} \sum_{k=1}^{\infty} \sum_{j=1}^{s} a_{js} k(k-1) \cdots (k-j+1) \frac{x^k}{k!} n^k$$

$$= \sum_{j=1}^{s} \frac{a_{js} x^j}{n^{s-j}} .$$

Since the a_{js} are positive and $\{1/n^i\}_{n=1}^{\infty}$ is a completely monotone sequence for all i, it follows that $\{M_n(t^s; x)\}$ is completely monotone.

Our next example is also quite easy. We define, using the Gauss kernel,

$$W_n(f; x) = \left(\frac{n}{\pi}\right)^{1/2} \int_{-\infty}^{\infty} f(t)\, e^{-n(t-x)^2}\, dt.$$

THEOREM 4. *Let f be an even function, absolutely monotone on* $[0, \infty)$, *such that*

$$\lim_{t\to\infty} |f(t)|/e^{\alpha t^2} = 0 \qquad \textit{for all} \quad \alpha > 0. \tag{21}$$

Then $\{W_n(f; x)\}_1^{\infty}$ is a completely monotone sequence for each fixed x.

Proof. Taking account of the growth condition (see [11, p. 438] and the extreme ray structure, it suffices to consider $\{W_n(t^{2s}; x)\}_{n=1}^{\infty}$, for $s = 1, 2, \ldots$. A straightforward computation yields

$$W_n(t^{2s}; x) = \left(\frac{n}{\pi}\right)^{1/2} \int_{-\infty}^{\infty} (t + x)^{2s} e^{-nt^2}\, dt$$

$$= \left(\frac{n}{\pi}\right)^{1/2} \sum_{i=0}^{s} \binom{2s}{2i} x^{2s-2i} \int_{-\infty}^{\infty} t^{2i} e^{-nt^2}\, dt$$

$$= \sum_{i=0}^{s} \binom{2s}{2i} x^{2s-2i} \frac{(2i)!}{2^{2i} n^i i!} .$$

Since $\{1/n^i\}_{n=1}^{\infty}$ is completely monotone for each i, so is this finite sum. Q.E.D.

As a final example, we discuss approximation operators of the type

$$U_n(f; x) = \int_0^{\infty} f\left(\frac{tx}{n}\right) \phi^{(n)}(t)\, dt, \qquad x > 0 \tag{22}$$

where $\phi(t)$ is the density function of a nonnegative random variable X with expectation 1, and $\phi^{(n)}(t)$ denotes the n-fold convolution of ϕ. We note that if $X_1, \ldots, X_n$ are independent random variables with density ϕ, then

$$U_n(f; x) = E\left[f\left(\frac{x \sum_{i=1}^{n} X_i}{n}\right)\right],$$

where $E[\cdot]$ is the expectation operator. Let

$$V(s) = \int_0^\infty e^{st}\phi(t)\, dt$$

and assume that

$$c = \sup\{s \mid V(s) < \infty\} > 0. \tag{23}$$

Let Γ be the class of continuous functions on $[0, \infty)$ obeying

$$|f(t)| < Ke^{(c-\alpha)t}, \qquad 0 \leqslant t \leqslant \infty,$$

for some positive constants K and α.

It is proved in ([6, p. 330]) that for each $f \in \Gamma$ we have $U_n(f; x) \to f(x)$, $0 \leqslant x < \infty$, where the convergence is uniform on compact subsets.

Assume now that ϕ is a Pólya Frequency (PF) density vanishing for $t < 0$. (For information concerning PF functions consult chapter 7 of [4]). The Laplace transform $\psi(s)$ of $\phi(t)$ has the form

$$\psi(s) = e^{-\delta s} \Big/ \prod_{i=1}^{\infty} (1 + \lambda_i s), \tag{24}$$

where $\delta \geqslant 0, \lambda_i > 0, \sum_{i=1}^{\infty} \lambda_i < \infty$ ([4, p. 345]). Hence $V(s) = e^{\delta s}/\prod_{i=1}^{\infty}(1 - \lambda_i s)$ and $c = \inf 1/\lambda_i$. Observe next that if the sequence $\{\lambda_i\}$ is infinite then $\lambda_i \to 0$. Thus $\inf 1/\lambda_i = \min 1/\lambda_i = c > 0$, so that (23) is satisfied.

We can now state

THEOREM 5. *Let $\phi(t)$ be a PF density vanishing for $t < 0$, whose Laplace transform is given in* (24). *Let $f \in \Gamma$, where $c = \min(1/\lambda_i) > 0$. Then $\{U_n(f; x)\}_{n=1}^{\infty}$ is completely monotone whenever f is absolutely monotone.*

Proof. We start with the basic PF density $\phi(\lambda, \delta; t) = \Theta(t)$ whose Laplace transform is given by

$$e^{-\delta s}/(1 + \lambda s), \qquad \delta > 0, \quad \lambda > 0.$$

Explicitly,

$$\begin{aligned} \theta(t) &= \frac{1}{\lambda} e^{-(t-\delta)/\lambda}, \qquad && t \geqslant \delta, \\ &= 0, && t < \delta. \end{aligned} \tag{25}$$

It is readily computed that

$$\begin{aligned} \theta^{(n)}(t) &= \frac{1}{\lambda^n} \exp[-(t - n\delta)/\lambda] \frac{(t - n\delta)^{n-1}}{(n-1)!}, \qquad && t \geqslant n\delta \\ &= 0, && t > n\delta \end{aligned} \tag{26}$$

Using this explicit expression, we have, for $r \geqslant 1$

$$\begin{aligned} U_n(t^r; x) &= \int_{n\delta}^{\infty} \frac{x^r t^r}{n^r} \frac{1}{\lambda^n} e^{-(t-n\delta)/\lambda} \frac{(t-n\delta)^{n-1}}{(n-1)!} \, dt \\ &= x^r \sum_{k=0}^{r} \frac{\delta^{r-k}}{n^k} \binom{r}{k} \lambda^k \frac{(k+n-1)!}{(n-1)!}. \end{aligned}$$

Note that

$$\begin{aligned} \frac{(k+n-1)!}{n^k (n-1)!} &= \left(1 + \frac{k-1}{n}\right)\left(1 + \frac{k-2}{n}\right) \cdots \left(1 + \frac{1}{n}\right) \\ &= \sum_{i=0}^{k-1} \frac{d_{k,i}}{n^i}, \qquad d_{k,i} > 0. \end{aligned}$$

Hence

$$U_n(t^r; x) = x^r \sum_{j=0}^{r-1} \frac{b_{j,r}(\theta)}{n^j}, \qquad b_{j,r}(\theta) > 0$$

This is a finite sum of completely monotone sequences; hence it is completely monotone.

Assume now that $\theta_1(t)$ and $\theta_2(t)$ are two basic PF-densities of the form (25), and let $\phi = \theta_1 * \theta_2$ be their convolution. Let $\{U_n\}_1^\infty$ be the operators corresponding to ϕ. Let X_i, $i = 1,..., n$ be independent identically distributed random variables with density θ_1, and let Y_i, $i = 1,..., n$ correspond to θ_2 in the same way. Then $X_i + Y_i$, $i = 1,..., n$ are independent random variables with density ϕ, and we have

$$\begin{aligned} U_n(t^r; x) &= x^r E \left\{\left(\frac{\sum_{i=1}^n (X_i + Y_i)}{n}\right)^r\right\} \\ &= x^r E \left[\left(\frac{\sum_{i=1}^n X_i}{n} + \frac{\sum_{i=1}^n Y_i}{n}\right)^r\right] \\ &= x^r \sum_{j=0}^{r} \binom{r}{j} E\left[\left(\frac{\sum_{i=1}^n X_i}{n}\right)^j\right] E\left[\left(\frac{\sum_{i=1}^n Y_i}{n}\right)^{r-j}\right] \\ &= x^r \sum_{j=0}^{r} \binom{r}{j} \left[\sum_{i=0}^{j} \frac{b_{i,j}(\theta_1)}{n^i}\right]\left[\sum_{i=0}^{r-j} \frac{b_{i,r-j}(\theta_2)}{n^i}\right] \\ &= x^r \sum_{k=0}^{r} \frac{b_{k,r}(\phi)}{n^k}. \end{aligned}$$

Since $b_{i,j}(\theta_1)$, $b_{i,j}(\theta_2)$ are nonnegative for all i, j so is also $b_{k,r}$ for all k, r. It follows that $\{U_n(t^r; x)\}_{n=1}^{\infty}$ is a completely monotone sequence.

Hence, if ϕ is a finite convolution of the basic PF-densities, then $U_n(t^r; x)$ is a finite positive linear combination of $\{n^{-i}\}_{i=0}^{r}$.

Note next that (cf. [4, p. 334]) if $\phi(t)$ is the general PF density, then $\phi(t) = \lim_{k+\infty} \phi_k(t)$, where the $\phi_k(t)$, $k = 1, 2,\ldots$, are finite convolutions of the basic PF-densities, and the convergence is uniform on compact subsets.

Hence for each fixed n, $\phi_k^{(n)}(t) \to \phi^{(n)}(t)$, and by the dominated convergence theorem

$$U_n(t^r; x)|_{\phi_k} \xrightarrow[k\to\infty]{} U_n(t^r; x)|_{\phi}, \qquad r = 0, 1,\ldots,$$

where the notation is obvious.

Since each of the expressions on the left is a finite positive linear combination of powers of $1/n^i$, $i \leqslant r$, the same result is valid for the limit.

Thus, $\{U_n(t^r; x)\}_{n=1}^{\infty}$ is a completely monotone sequence, and this suffices, by the previously employed arguments, to establish the theorem. Q.E.D.

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